MARMARA UNIVERSITY
COMPUTER SCIENCE ENGINEERING
LINEAR ALGEBRA
FINAL HOMEWORK
22/06/2020

Write your solutions on a standard white paper. You should write your name, last name, student number, the declaration of truth (DOĞRULUK BEYANI: Bu sınav kağıdındaki tüm sorular tarafımca çözülmüş olup, hiçbir yolla başka bir şahıstan yardım alınmamış ve başka bir şahısla paylaşılmamıştır.) and your signature on each paper. Scan your solution papers possibly with your phone and upload it to ues.marmara.edu.tr as a single pdf file with file name name_lastname before 13:00. If you encounter upload problems, you can send your solution pdf file to the email address taylan.sengul@marmara.edu.tr. Each problem is worth 20pts. Show your solution steps clearly to get credit. Good luck!

1. Are the following subsets subspaces of $\mathbb{R}^3$? If so, find their basis. If not, explain the reason.
(A) $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : xyz=0\right\}$, (B) $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : x \le y \le z\right\}$, (C) $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : x=y\right\}$, (D) $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : x=1\right\}$.

2. For what values of h, do the vectors $\begin{pmatrix}4\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}, \begin{pmatrix}-6\\3\\h\end{pmatrix}$ span $\mathbb{R}^3$?

3. Is there a linear transformation $T : \mathbb{R}^2 \mapsto P_3$ which maps $T\begin{pmatrix}1\\2\end{pmatrix} = x^3 - 1$, and $T\begin{pmatrix}-2\\1\end{pmatrix} = x^2 + x + 1$? If there is, find $T\begin{pmatrix}7\\4\end{pmatrix}$.

4. Let
$$A = \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$$
Find a basis for the solution space of $A\mathbf{x} = \mathbf{0}$.

5. (A) Let A be a square matrix. If λ_1 is an eigenvalue of A with eigenvector v_1, and λ_2 is an eigenvalue of A with eigenvector v_2, show that $A^n(c_1v_1 + c_2v_2) = c_1\lambda_1^n v_1 + c_2\lambda_2^n v_2$ for any positive integer n and any scalars c_1, c_2. (Here A^n is the matrix A multiplied n times with itself.)
(B) Suppose $\lambda_1 = 2$, $v_1 = \begin{pmatrix}1\\1\end{pmatrix}$, $\lambda_2 = 1$ and $v_2 = \begin{pmatrix}2\\3\end{pmatrix}$ above. Find $A^{100}\begin{pmatrix}3\\4\end{pmatrix}$.

1 **(A) No:**

Reason:

For subset A to be a subspace,
* we take u, v from A, $v+u$ must be included in A which is not this case because if we take:

$$u = \begin{pmatrix}1\\0\\1\end{pmatrix} \in A, \quad v = \begin{pmatrix}0\\1\\0\end{pmatrix} \in A$$

But: $u + v = \begin{pmatrix}1\\1\\1\end{pmatrix} \notin A$ because $1.1.1 \neq 0$

thus this subset is not a subspace of $\mathbb{R}^3$.

(B) No:

Reason:

- The second property for subspaces is not applicable for this subset.
- → Take $c < 0$ ($c \in \mathbb{R}^-$)

$$c\begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \overset{x'}{\widehat{cx}} \geq \overset{y'}{\widehat{cy}} \geq \overset{z'}{\widehat{cz}}$$

which contradicts with $x' < y' \leq z'$

thus $\begin{pmatrix}cx\\cy\\cz\end{pmatrix} \notin B$

so this subset is not a subspace of $\mathbb{R}^3$

(C) Yes: $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : x=y\right\}$

$$\left\{\begin{pmatrix}x\\x\\z\end{pmatrix}\right\} = \left\{x\begin{pmatrix}1\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \quad (*)$$

So: $\boxed{\text{basis } S = \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}}$

→ Show that S is a basis:

1) $a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}a\\a\\b\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \boxed{a=b=0}$$

⇒ the vectors are lin. indep.

2) From (*) we can deduce directly that they spane the subspace. So, they form a basis.

(D) No.

Reason:

Take $u = \begin{pmatrix}x\\y\\z\end{pmatrix} \in D$

$v = \begin{pmatrix}x'\\y'\\z'\end{pmatrix} \in D$

$$u+v = \begin{pmatrix}1\\y\\z\end{pmatrix} + \begin{pmatrix}1\\y'\\z'\end{pmatrix} = \begin{pmatrix}2\\y+y'\\z+z'\end{pmatrix} \notin D$$

⇒ D is not a subspace of $\mathbb{R}^3$.

Note: In (C) we can check that it is a subspace by checking:

1) take: $u = \begin{pmatrix}x\\x\\z\end{pmatrix}$ $\quad v = \begin{pmatrix}x'\\x'\\z'\end{pmatrix}$

$u + v = \begin{pmatrix}x+x'\\x+x'\\z+z'\end{pmatrix} \in C$

2) if $d \in \mathbb{R}$ and $u = \begin{pmatrix}x\\x\\z\end{pmatrix}$

$du = \begin{pmatrix}dx\\dx\\dz\end{pmatrix} \in C$

2 The vectors span $\mathbb{R}^3$ means that:

$$a\begin{pmatrix}4\\1\\1\end{pmatrix}+b\begin{pmatrix}2\\2\\0\end{pmatrix}+c\begin{pmatrix}-6\\3\\h\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\left[\begin{array}{ccc|c}4&2&-6&x\\1&2&3&y\\1&0&h&z\end{array}\right]$$

$r_2 \leftrightarrow r_1$

$$\left[\begin{array}{ccc|c}1&2&3&y\\4&2&-6&x\\1&0&h&z\end{array}\right]$$

$-4r_1+r_2 \to r_2$
$-r_1+r_3 \to r_3$

$$\left[\begin{array}{ccc|c}1&2&3&y\\0&-6&-18&x-4y\\0&-2&h-3&z-x\end{array}\right]$$

$-\frac{1}{3}r_2+r_3 \to r_3$
$-\frac{1}{6}r_2 \to r_2$

$$\left[\begin{array}{ccc|c}1&2&3&y\\0&1&3&\frac{-x+4y}{6}\\0&0&h+3&z-x-\left(\frac{x-4y}{3}\right)\end{array}\right]$$

. For the vectors to span $\mathbb{R}^3$

$h+3 \neq 0$

$\boxed{h \neq -3}$

$h \in \left]-\infty,-3\right[\cup \left]-3,+\infty\right[$

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$$A=\begin{bmatrix}1&-10&-24&-42\\1&-8&-18&-32\\-2&20&51&87\end{bmatrix}$$

$-r_1+r_2 \to r_2$
$2r_1+r_3 \to r_3$

$$\left[\begin{array}{cccc|c}1&-10&-24&-42&0\\0&2&6&10&0\\0&0&3&3&0\end{array}\right]$$

$\frac{1}{2}r_2 \to r_2$; $\frac{1}{3}r_3 \to r_3$

$$\left[\begin{array}{cccc|c}1&-10&-24&-42&0\\0&1&3&5&0\\0&0&1&1&0\end{array}\right]$$

$-3r_3+r_2 \to r_2$

$24r_3+r_1 \to r_1$

$$\left[\begin{array}{cccc|c}1&-10&0&-18&0\\0&1&0&2&0\\0&0&1&1&0\end{array}\right]$$

$10r_2+r_1 \to r_1$

$$\left[\begin{array}{cccc|c}1&0&0&2&0\\0&1&0&2&0\\0&0&1&1&0\end{array}\right]$$

put $\boxed{x_4 = r}$

$x_3 + x_4 = 0 \Rightarrow \boxed{x_3 = -r}$

$x_2 + 2x_4 = 0 \Rightarrow \boxed{x_2 = -2r}$

$x_1 + 2x_4 = 0 \Rightarrow \boxed{x_1 = -2r}$

so; $\vec{x} = \begin{bmatrix} -r \\ -2r \\ -2r \end{bmatrix} = r \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}$

so : $\left\{ \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} \right\}$ forms a basis for the soln space.

3 suppose T is a Linear Transformation

so:

$T\left[\begin{pmatrix} a \\ b \end{pmatrix}\right] = ?$

$\begin{bmatrix} a \\ b \end{bmatrix} = m \begin{bmatrix} 1 \\ 2 \end{bmatrix} + n \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & a \\ 2 & 1 & b \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & a \\ 0 & 5 & b-2a \end{array}\right]$$

$\Rightarrow n = \dfrac{b-2a}{5}$

$m = \dfrac{a+2b}{5}$

so : $T\left[\begin{pmatrix} a \\ b \end{pmatrix}\right] = T\left[m\begin{pmatrix} 1 \\ 2 \end{pmatrix} + n \begin{pmatrix} -2 \\ 1 \end{pmatrix}\right]$

$= m\, T\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right] + n\, T\left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right]$

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$T\left[\begin{pmatrix} a \\ b \end{pmatrix}\right] = \dfrac{a+2b}{5}(x^3-1) + \dfrac{b-2a}{5}(x^2+x+1)$

$\Rightarrow$ Check if it is indeed a lin Trans:

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$T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} a' \\ b' \end{bmatrix}\right) = T\left(\begin{bmatrix} a+a' \\ b+b' \end{bmatrix}\right)$

$= \dfrac{(a+a') + 2(b+b')}{5}(x^3-1) + \dfrac{(b+b') - 2(a+a')}{5}(x^2+x+1)$

$= \dfrac{a+2b}{5}(x^3-1) + \dfrac{a'+2b'}{5}(x^3-1)$

$+ \dfrac{b-2a}{5}(x^2+x+1) + \dfrac{b'-2a'}{5}(x^2+x+1)$

$= T\left[\begin{pmatrix} a \\ b \end{pmatrix}\right] + T\left[\begin{pmatrix} a' \\ b' \end{pmatrix}\right]$

2 $c \in \mathbb{R}$

$T\left(c\begin{bmatrix} a \\ b \end{bmatrix}\right) = c\left(\dfrac{a+2b}{5}(x^3-1) + \dfrac{b-2a}{5}(x^2+x+1)\right)$

$= c\, T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$

Thus; there exists a linear transformation $T: \mathbb{R}^2 \longrightarrow P_3$ which maps the vectors in the question

$T\left[\begin{pmatrix} 7 \\ 4 \end{pmatrix}\right] = \dfrac{7+8}{5}(x^3-1) + \dfrac{4-14}{5}(x^2+x+1)$

$= 3(x^3-1) - 2(x^2+x+1)$

$T\left[\begin{pmatrix} 7 \\ 4 \end{pmatrix}\right] = 3x^3 - 2x^2 - 2x - 5$

$\boxed{5}$ (A) we know that for eigenvectors v_1 and v_2:

$$Av_1 = \lambda_1 v_1 \quad / \quad Av_2 = \lambda_2 v_2$$

Thus:

$$A^n(c_1v_1 + c_2v_2) = c_1 A^n v_1 + c_2 A^n v_2$$

$$= c_1 A^{n-1}(Av_1) + c_2 A^{n-1}(Av_2)$$

$$= c_1 A^{n-1}(\lambda_1 v_1) + c_2 A^{n-1}(\lambda_2 v_2)$$

$$= c_1 \lambda_1 A^{n-2}(Av_1) + c_2 \lambda_2 A^{n-2}(Av_2)$$

$$= c_1 \lambda_1 A^{n-2}(\lambda_1 v_1) + c_2 \lambda_2 A^{n-2}(\lambda_2 v_2)$$

$$= c_1 \lambda_1^2 A^{n-2} v_1 + c_2 \lambda_2^2 A^{n-2} v_2$$

we continue this way

$$= c_1 \lambda_1^n \underbrace{A^{n-n}}_{I_n} v_1 + c_2 \lambda_2^n \underbrace{A^{n-n}}_{I_n} v_2$$

$$A^n(c_1v_1 + c_2v_2) = c_1 \lambda_1^n v_1 + c_2 \lambda_2^n v_2$$

(B)

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

we can diretly notice that:

$$\boxed{c_1 = c_2 = 1}$$

$$A^{100}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = A^{100}\left(1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)$$

According to the equation in question (A):

$$A^{100}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1 \cdot (2)^{100}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1 \cdot (1)^{100}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$A^{100}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{bmatrix} 2^{100} + 2 \\ 2^{100} + 3 \end{bmatrix}$$